

Secondary Spectrum Auctions for Symmetric and Submodular Bidders

Martin Hoefer*

Thomas Kesselheim†

Abstract

We study truthful auctions for secondary spectrum usage in wireless networks. In this scenario, n communication requests need to be allocated to k available channels that are subject to interference and noise. We present the first truthful mechanisms for secondary spectrum auctions with symmetric or submodular valuations. Our approach to model interference uses an edge-weighted conflict graph, and our algorithms provide asymptotically almost optimal approximation bounds for conflict graphs with a small inductive independence number $\rho \ll n$. This approach covers a large variety of interference models such as, e.g., the protocol model or the recently popular physical model of interference. For unweighted conflict graphs and symmetric valuations we use LP-rounding to obtain $O(\rho)$ -approximate mechanisms; for weighted conflict graphs we get a factor of $O(\rho \cdot (\log n + \log k))$. For submodular users we combine the convex rounding framework of [12] with randomized meta-rounding to obtain $O(\rho)$ -approximate mechanisms for matroid-rank-sum valuations; for weighted conflict graphs we can fully drop the dependence on k to get $O(\rho \cdot \log n)$. We conclude with promising initial results for deterministically truthful mechanisms that allow approximation factors based on ρ .

*Department of Computer Science, RWTH Aachen University, mhoefer@cs.rwth-aachen.de. Supported by DFG grant Ho 3831/3-1.

†Department of Computer Science, RWTH Aachen University, kesselheim@cs.rwth-aachen.de. Supported by DFG through UMIC Research Centre at RWTH Aachen University.

1 Introduction

The development of wireless networks crucially relies on successful management of the frequency spectrum to provide reliable network access. Nowadays, spectrum allocation is static – service providers (so-called *primary users*) can obtain nation-wide licenses for channels in governmental spectrum auctions. This practice is inefficient and problematic: While primary users often use their spectrum bands only in selected local areas, new and innovative applications suffer in their development, because global licenses are difficult to obtain or generally unavailable. A major research effort is currently underway in computer science and engineering to overcome this artificial scarcity and let primary users open their bands in local areas for so-called secondary usage. *Auctions* are attractive to coordinate secondary spectrum usage, as they allow implementing social or monetary goals in a market with self-interested participants having private information. Interest in secondary spectrum auctions has increased significantly in recent years (see [15, 16, 20, 35, 36], and [2] for a general discussion), but the algorithmic and strategic problems are still poorly understood.

In secondary spectrum markets, a natural regulatory goal is to maximize social welfare, i.e., the total valuation or benefit of the channel allocation to the secondary users. As constraint for the allocation, the assigned channels must allow successful transmission in the presence of interference and noise. Positioning and interference situation is often known or can sometimes even be observed publicly, but valuations are private information of the users and have to be collected by the algorithm. In this process, secondary users have an obvious incentive to manipulate the algorithm by misreporting their valuation. In this paper, we therefore strive to design *truthful mechanisms* that allocate channels and use payments to motivate users to reveal their values truthfully.

This scenario represents a novel and non-trivial extension of combinatorial auctions. In combinatorial auctions we have to allocate k indivisible items (channels) to n bidders (users). Each bidder v has a valuation $b_v(S)$ for any subset S of items. The goal is to maximize social welfare, i.e., the sum of (reported) valuations for the assigned item sets. Secondary spectrum auctions extend this model by allowing to give a single item/channel to *multiple users* if the set of users is feasible in terms of interference. Interference can be modeled in various ways, and we follow the approach of [20] where users are vertices in a publicly known edge-weighted conflict graph. A set of users is feasible for a channel if they form an independent set in the graph, for a suitably defined notion of independent set. This approach covers virtually all existing interference models in the literature [20, 33]. For instance, if users are communication requests in the physical model of interference, we can use edge weights corresponding to the affectance between requests, and feasibility due to bounded signal-to-interference-plus-noise ratio (SINR) is then equivalent to having an independent set (as defined below, see also [20]).

Interestingly, conflict graphs resulting from popular interference models (e.g., protocol model [32] or physical model [20–22]) have a small *inductive independence number* ρ . The wide applicability of this non-standard graph parameter for algorithm design is only recently starting to be explored [1, 6, 34]. For our secondary spectrum auctions it allows to bypass well-known lower bounds of $\Omega(n^{1-\epsilon})$ for approximating independent set and derive significantly improved guarantees based on ρ [20]. However, even in ordinary combinatorial auctions with $\rho = 1$ any efficient algorithm can only achieve a factor of essentially $\min\{n, \sqrt{k}\}$ unless we make additional assumptions on the user valuations [25, 26].

1.1 Contribution

In this paper, we design randomized auctions for spectrum markets, where secondary users strive to acquire one or more of a set of available channels. Making additional assumptions on the user valuations allows us to bypass the $\min\{n, \sqrt{k}\}$ lower bound and to significantly improve previous results. We examine the two prominent classes of symmetric and submodular valuations. Both classes occupy a central position in the literature on combination auctions, and they have very natural and intuitive interpretations in the context of secondary spectrum auctions.

Symmetric valuations are the analog to multi-unit auctions, where each valuation only depends on the *number* of channels rather than the exact subset. This is a natural assumption in a secondary spectrum auction of equally sized channels which all offer very similar conditions. Submodularity is economically interpreted as diminishing marginal returns. A common representative are coverage valuations, where users pick elements each covering a certain range, and the value is the total covered area. This is a natural assumption, e.g, when secondary users are transmitters that strive to be received by as many mobile stations as possible, where each of the latter operates on a fixed subset of channels.

For symmetric valuations (see Section 3) we use the intuition of multi-unit auctions and round a suitably defined linear program yielding only an assignment of numbers of channels. Using these numbers an independent set for each channel is then created by a greedy approach. This allows to avoid dependence on k and obtain an approximation factor of $O(\rho)$ for unweighted conflict graphs. Note that this is asymptotically almost optimal under standard complexity assumptions. Theorem 5 in [20] shows that there is no $\rho/2^{O(\sqrt{\log \rho})}$ -approximation unless $P = NP$. Truthfulness is achieved via combination of our approach with the celebrated randomized meta-rounding framework by Lavi and Swamy [24]. For edge-weighted conflict graphs, the construction step of independent sets is significantly more involved. The asymmetry of conflicts inherent in edge-weighted graphs require the use of additional concurrent contention resolution methods to partition the rounded set of requests into feasible independent sets. This approach allows to obtain a factor of $O(\rho \cdot (\log n + \log k))$. Our resulting mechanisms are randomized, run in polynomial time, and yield truthfulness in expectation.

For submodular valuations (see Section 4) we focus on matroid-rank-sum valuations, which encompass the most frequently studied submodular valuations. We design randomized mechanisms that fall into the class of maximum-in-distributional range (MIDR) mechanisms. In particular, our approach is along the lines of the convex rounding technique recently pioneered in [11, 12] and achieves an approximation factor of $O(\rho)$ for unweighted conflict graphs. Again, this is asymptotically almost optimal under standard complexity assumptions. In contrast to the case of symmetric valuations, we can fully omit the dependence on k and show factors of $O(\rho \cdot \log n)$ even for weighted conflict graphs. Our rounding scheme is similar to the Poisson rounding scheme from [12]. The main difference and complication is again the need to round each channel to an independent set of users. To achieve this, we round independently for each channel and build the required support of independent sets using a randomized meta-rounding approach. Probably the most technical contribution is showing that this rounding scheme preserves the favorable conditioning properties that allow to apply convex optimization techniques to compute the underlying distribution with sufficient precision in expected polynomial time, even for weighted conflict graphs. Our resulting mechanisms are again randomized and provide truthfulness in expectation.

Finally, we also briefly discuss designing deterministic truthful mechanisms (see Section 5). We present a promising initial result, a monotone greedy $O(\rho \cdot \log n)$ -algorithm for a single channel in unweighted conflict graphs. However, this area remains mostly as an interesting and important

avenue for future work.

1.2 Related Work

Our paper is connected to recent approaches for designing truthful mechanisms in secondary spectrum markets without [35, 36] and with non-trivial worst-case approximation guarantees, e.g., for social welfare and fairness [16] or revenue [15]. However, all these works are restricted to a single channel and unweighted conflict graphs. To this date the only general (analytical) treatment of approximation algorithms and truthful mechanisms for multi-channel secondary spectrum auctions is [20] where truthful-in-expectation mechanisms for general user valuations are designed using the inductive independence number in edge-weighted conflict graphs. For unweighted conflict graphs the approximation guarantee is $O(\rho \cdot \sqrt{k})$, for edge-weighted conflict graphs $O(\rho \cdot \sqrt{k} \cdot \log n)$. The former result is asymptotically almost optimal in ρ if $k = 1$ [29] and in k if $\rho = 1$. The latter lower bound is a well-known result in combinatorial auctions [25, 26].

In ordinary combinatorial auctions, these strong lower bounds initiated the study of relevant subclasses of valuations, for an overview see, e.g., [3]. Symmetric valuations essentially pose a knapsack problem of assigning numbers of items to bidders, and a deterministic truthful greedy 2-approximation [27] was the first benchmark solution. Since then there has been significant progress including, e.g., approximation schemes for single-minded bidders [4], k -minded bidders [10], or monotone valuations [8, 30]. In contrast to these works, we must additionally decompose assigned numbers of channels into an independent set for each single channel. Here we rely on rounding linear programs to ensure that such a decomposition exists and can be found in polynomial time.

For submodular valuations, social welfare maximization without truthfulness is essentially solved. Optimal $(1 - 1/e)$ -approximation algorithms exist even for value oracle access [31], where each valuation b_v is an oracle that we can query to obtain $b_v(S)$ for a single set S in each operation. This factor cannot be improved assuming either polynomial communication [26] in the value oracle model or polynomial-time complexity in general [23]. For the strategic setting and general submodular valuations, the best factors are $O\left(\frac{\log m}{\log \log m}\right)$ for truthfulness in expectation [9], and $O(\log m \log \log m)$ for universal truthfulness [7]. Dughmi et al [12] recently proposed a convex rounding technique to build truthful-in-expectation mechanisms. Their approach yields an optimal $(1 - 1/e)$ -approximation for the class of matroid-rank-sum valuations. It follows the idea of maximal-in-distributional range (MIDR) mechanisms by defining a range of distributions independent of the valuations and a rounding procedure. Both are designed in a way that finding the optimal distribution over the range for the reported valuations becomes a convex program with favorable conditioning properties. Hence, the optimal distribution can be found using suitable convex optimization methods in expected polynomial time. Truthfulness follows using the Vickrey-Clarke-Groves (VCG) payment scheme. Very recently, Dughmi and Vondrak showed that a similar result cannot be obtained for general submodular valuations in the oracle model [13].

Designing (non-truthful) algorithms for independent set problems in conflict graphs has received significant attention recently, especially for graphs based on the physical model of interference with SINR constraints. If each request has a value of 1 for being in the independent set, asymptotically optimal performance bounds for specific transmission power assignments were obtained when requests are located in various classes of metric spaces [14, 17, 19]. For the problem where powers can be arbitrarily chosen, there is a constant-factor approximation algorithm [21].

The inductive independence number is a non-standard graph parameter that is only recently

starting to receive increased attention. Up to our knowledge the parameter has first been used in [1], and since then has been rediscovered independently a number of times (see, e.g., [32]). Ye and Borodin [34] recently conducted the first study addressing general issues that arise when using the measure for solving algorithmic problems in unweighted graphs. The eminent usefulness of the parameter for analyzing interference models and spectrum markets was highlighted in [20].

2 Preliminaries

2.1 Channel Allocation in Spectrum Markets

In secondary spectrum markets there is a set $[k]$ of k available *channels* and a set V of n *users* or *bidders*. Each user $v \in V$ has a *valuation* or *benefit* $b_v : 2^{[k]} \rightarrow \mathbb{R}^+$. A valuation function b_v is called *symmetric* if $b_v(T) = b_v(|T|)$ for all $T \subseteq [k]$. It is *submodular* if $b_v(T \cup T') + b_v(T \cap T') \leq b_v(T) + b_v(T')$ for all $T \subseteq T'$. For submodular valuations we also assume they are *monotone* with $b_v(T) \leq b_v(T')$ for $T \subseteq T'$. A valuation b_v is a *matroid rank sum (MRS)* function if there exists a family of matroid rank functions $u_1, \dots, u_\kappa : 2^{[k]} \rightarrow \mathbb{N}$, and associated non-negative weights $w_1, \dots, w_\kappa \in \mathbb{R}^+$, such that $b_v(T) = \sum_{\ell=1}^\kappa w_\ell u_\ell(T)$ for all $T \subseteq [k]$.

To model interference we represent users as vertices in a complete edge-weighted and directed *conflict graph* $G = (V, E, w)$. The weight $w(u, v)$ of edge (u, v) represents the interference that user u creates for user v if both are assigned to the same channel. Interference between users is similar on each channel. A set of users $U \subseteq V$ is *feasible* or an *independent set* if $\sum_{u \in U} w(u, v) < 1$ for all $v \in U$. In *unweighted* conflict graphs all weights $w(u, v) \in \{0, 1\}$ and our definition of independent set is the same as in the classical sense. For many standard interference models, we can define weighted conflict graphs such that independent sets are exactly the sets for which we can have successful simultaneous transmission in the interference model. For instance, the protocol model results in unweighted conflict graphs, or the physical model of interference yields weighted conflict graphs where independent sets are feasible with respect to the SINR; for details see [20].

The algorithmic challenge in secondary spectrum markets is the *channel allocation problem*. In an optimal solution S , each user v receives a subset of channels $S_v \subseteq [k]$ such that each channel is given to an independent set in the conflict graph and the *social welfare* $b(S) = \sum_{v \in V} b_v(S_v)$ is maximized. In contrast to ordinary combinatorial auctions, an independent set can include more than one user. Our mechanisms cope with this issue using a structural parameter called inductive independence number. Let us define *symmetric weights* by $\bar{w}(u, v) = w(u, v) + w(v, u)$. Then the *inductive independence number* is the smallest number ρ such that there is an ordering π of the vertices satisfying the following condition: For all $v \in V$ and all independent sets $M \subseteq V$, we let $M_v = M \cap \{u \in V \mid \pi(u) < \pi(v)\}$ and have that $\sum_{u \in M_v} \bar{w}(u, v) \leq \rho$. Hence, ρ is the smallest number such that by picking the best ordering we can bound for any $v \in V$ the incoming weight from any independent set among previous vertices to at most ρ . We assume that ρ and the ordering π of V are given. For many interference models and their resulting conflict graphs we can find in polynomial time small upper bounds on ρ and a corresponding ordering witnessing ρ . For example, in the protocol model $\rho = O(1)$ [32] and in the physical model $\rho = O(\log n)$ [22] or $\rho = O(1)$ [21], depending on power control assumptions. In both cases, π orders users with decreasing or increasing distance between sender and receiver.

2.2 Mechanism Design Basics

To avoid that user v will strategically misreport his valuation, we charge payments p_v and make *truthfulness* a dominant strategy. For each user $v \in V$ we ensure that his *quasi-linear utility* satisfies $b_v(S_v) - p_v(b_v, b_{-v}) \geq b_v(S'(v)) - p_v(b'_v, b_{-v})$, where S and S' are our solutions to the channel allocation problem when v reports the true b_v and a some possibly other b'_v , respectively. This can be achieved using classic *Vickrey-Clarke-Groves (VCG) payments* if the allocation problem is always solved optimally.

In contrast, efficient truthful mechanisms cannot compute optimal solutions to intractable problems. For some problems, deterministic mechanisms can achieve only trivial approximation guarantees [28]. The situation is much better if we resort to *randomized mechanisms*, which define a distribution D over the set of solutions \mathcal{S} for the channel allocation problem and output an allocation $S \in \mathcal{S}$ according to D . In this case, we aim for *truthfulness in expectation*, i.e., for every $v \in V$

$$\mathbf{E}_{S \sim D} [b_v(S_v) - p_v(b_v, b_{-v})] \geq \mathbf{E}_{S \sim D'} [b_v(S_v) - p_v(b'_v, b_{-v})] ,$$

where D' is the distribution if v reports b'_v instead of b_v . A general technique to design such mechanisms is *maximal-in-distributional range (MIDR)*. Here we fix a set (the range) of distributions \mathcal{D} over \mathcal{S} , where \mathcal{D} is independent of the valuations b_v . The algorithm receives all reported valuations b_v and optimizes exactly over \mathcal{D} to find $D \in \mathcal{D}$ with maximum expected social welfare. Due to exact optimization over \mathcal{D} , the mechanism can use VCG payments to guarantee truthfulness in expectation. The obvious problem in MIDR is designing the distributional range \mathcal{D} (1) large enough to contain a good approximation for every possible vector of user valuations, and (2) small enough to allow for exact optimization over \mathcal{D} in polynomial time. Our mechanisms in Sections 3 and 4 will all be MIDR mechanisms. In Section 5 we also briefly treat designing greedy mechanisms that are truthful and deterministic.

3 Symmetric Valuations

In this section we consider spectrum auctions with *symmetric valuations* in which $b_v(T) = b_v(|T|)$ for all $v \in V$. We concentrate on designing approximation algorithms that can be turned into truthful MIDR mechanisms following the framework by Lavi and Swamy [24].

Our algorithms round the following LP relaxation based on $k \cdot |V|$ variables $x_{v,i} \in \{0, 1\}$ indicating if v gets exactly i channels or not. The relaxation reads

$$\begin{aligned} \text{Max.} \quad & \sum_{v \in V} \sum_{i=1}^k b_v(i) \cdot x_{v,i} \\ \text{s.t.} \quad & \sum_{\substack{u \in V \\ \pi(u) < \pi(v)}} \sum_{i=1}^k i \cdot \bar{w}(u, v) \cdot x_{u,i} \leq \rho \cdot k \quad \text{for all } v \in V \\ & \sum_{i=1}^k x_{v,i} \leq 1 \quad \text{for all } v \in V \\ & x_{v,i} \geq 0 \quad \text{for all } v \in V, i \in [k]. \end{aligned} \tag{1}$$

Note that this relaxation does not describe the problem exactly, as an integral solution to the relaxation might not be feasible for the channel allocation problem. In particular, the relaxation

Algorithm 1: LP-Rounding for Symmetric Valuations and Unweighted Conflict Graphs

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1 Decompose an optimal solution  $x$  to LP (1) into two solutions  $x^{(1)}$  and  $x^{(2)}$  as follows:
   Set  $x_{v,i}^{(1)} = x_{v,i}$  if  $i \leq k/2$  and  $x_{v,i}^{(1)} = 0$  otherwise; set  $x^{(2)} = x - x^{(1)}$ .
2 for  $l \in \{1, 2\}$  do
3   for  $v \in V$  in increasing order of  $\pi$  values do
4     with probability  $\frac{x_{v,i}^{(l)}}{4\rho}$  set  $d_v^{(l)} := i$ 
5     Let  $F_v^{(l)} := \{i \in [k] \mid \text{there is no } u \in \Gamma_\pi(v) \text{ with } i \in S_v^{(l)}\}$ 
6      $S_v^{(l)} = \begin{cases} \text{arbitrary } M \subseteq F_v^{(l)} \text{ with } |M| = d_v^{(l)} & \text{if } |F_v^{(l)}| \geq d_v^{(l)}, \\ \emptyset & \text{otherwise} \end{cases}$ 
7 Return the better one of the solutions  $S^{(1)}$  and  $S^{(2)}$ 

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does not specify which user receives which channel, but this information is critical for interference and feasibility of the requests.

We solve the LP relaxation optimally. The computed fractional solution is then decomposed into two solutions $x^{(1)}$ and $x^{(2)}$, that are rounded separately. Based on such a solution, for each user v a preliminary number of channels $d_v^{(l)}$ is determined at random. The probability is proportional to the fractional variables $x_{v,i}^{(l)}$. Having assigned these numbers of channels, we still have to derive a feasible allocation. In this allocation, each user v either gets $d_v^{(l)}$ channels or none.

3.1 Unweighted Conflict Graphs

In the case of unweighted conflict graphs, we use a simple greedy approach to distribute available channels to users, see Algorithm 1. The expected social welfare of the output will decrease only by a factor of $O(\rho)$ under the fractional optimum, which is asymptotically optimal.

Theorem 1. *Algorithm 1 returns a feasible allocation of social welfare at least $b^*/16\rho$ in expectation.*

Proof. Solutions $S^{(1)}$, $S^{(2)}$ separate the problem into two subproblems, in which the maximum or minimum non-zero number of channels allocated to a single player is $k/2$, respectively. We analyze both of these cases separately in the key proposition.

Proposition 2. *For $l \in \{1, 2\}$ and the expected social welfare of $S^{(l)}$ we have*

$$\mathbf{E} [b(S^{(l)})] \geq \frac{1}{8\rho} \cdot \sum_{v \in V} \sum_{i=1}^k b_v(i) \cdot x_{v,i}^{(l)}.$$

Proof. For all $v \in V$, $i \in [k]$, $l \in \{0, 1\}$ let $X_{v,i}^{(l)}$ be a 0/1 random variable indicating if in the rounding stage $d_v^{(l)}$ is set to i . We know that $\mathbf{Pr} [X_{v,i}^{(l)} = 1] = \frac{x_{v,i}^{(l)}}{4\rho}$. Let $Y_{v,i}^{(l)}$ be a 0/1 random variable indicating if $|S_v^{(l)}| = i$. To show the proposition it remains to bound $\mathbf{Pr} [Y_{v,i}^{(l)} = 0 \mid X_{v,i}^{(l)} = 1]$; that is, the probability that a user v does not receive i channels although $d_v^{(l)}$ was set to i .

Case $l = 1$: The event that $Y_{v,i}^{(1)} = 0$ but $X_{v,i}^{(1)} = 1$ can only occur if $|F_v^{(1)}| \leq i$. So in particular $|F_v^{(1)}| \leq k/2$. We can express $|F_v^{(1)}|$ in terms of $Y_{v,i}^{(1)}$ as

$$k - |F_v^{(1)}| \leq \sum_{u \in \Gamma_\pi(v)} \sum_{i=1}^k i \cdot Y_{u,i}^{(1)} \leq \sum_{u \in \Gamma_\pi(v)} \sum_{i=1}^k i \cdot X_{u,i}^{(1)} .$$

By linearity of expectation and the definition of ρ this yields

$$\mathbf{E} \left[k - |F_v^{(1)}| \right] \leq \sum_{u \in \Gamma_\pi(v)} \sum_{i=1}^k i \cdot \frac{x_{u,i}^{(1)}}{4\rho} \leq \frac{k}{4} .$$

So, we get by Markov inequality

$$\mathbf{Pr} \left[|F_v^{(1)}| \leq \frac{k}{2} \right] = \mathbf{Pr} \left[k - |F_v^{(1)}| \geq \frac{k}{2} \right] \leq \frac{1}{2} .$$

In total this yields

$$\mathbf{Pr} \left[Y_{v,i}^{(1)} = 1 \right] = \mathbf{Pr} \left[X_{v,i}^{(1)} = 1 \right] \cdot \mathbf{Pr} \left[|F_v^{(1)}| \geq i \right] \geq \mathbf{Pr} \left[X_{v,i}^{(1)} = 1 \right] \cdot \mathbf{Pr} \left[|F_v^{(1)}| \geq \frac{k}{2} \right] \geq \frac{x_{v,i}^{(1)}}{8\rho} ,$$

which proves the proposition in Case 1.

Case $l = 2$: The event that $Y_{v,i}^{(2)} = 0$ but $X_{v,i}^{(2)} = 1$ can only happen if there is some $u \in \Gamma_\pi(v)$

with $S_u^{(2)} \neq \emptyset$, in which case $\sum_{u \in \Gamma_\pi(v)} \sum_{i=k/2+1}^k Y_{u,i}^{(2)} \geq 1$. Furthermore, we have

$$\sum_{u \in \Gamma_\pi(v)} \sum_{i=k/2+1}^k Y_{u,i}^{(2)} \leq \sum_{u \in \Gamma_\pi(v)} \sum_{i=k/2+1}^k X_{u,i}^{(2)} \leq \frac{2}{k} \sum_{u \in \Gamma_\pi(v)} \sum_{i=k/2+1}^k i \cdot X_{u,i}^{(2)}$$

Using linearity of expectation and the definition of ρ this yields

$$\mathbf{E} \left[\sum_{u \in \Gamma_\pi(v)} \sum_{i=k/2+1}^k Y_{u,i}^{(2)} \right] \leq \frac{2}{k} \sum_{u \in \Gamma_\pi(v)} \sum_{i=k/2+1}^k i \cdot \frac{x_{u,i}^{(2)}}{4\rho} \leq \frac{2}{k} \cdot \frac{k}{4} \leq \frac{1}{2} ,$$

Markov inequality then implies that the probability that all of the $u \in \Gamma_\pi(v)$ have $S_u^{(2)} = \emptyset$ is at least $1/2$. This means

$$\begin{aligned} \mathbf{Pr} \left[Y_{v,i}^{(2)} = 1 \right] &= \mathbf{Pr} \left[X_{v,i}^{(2)} = 1 \right] \cdot \mathbf{Pr} \left[|F_v^{(1)}| \geq i \right] \\ &\geq \mathbf{Pr} \left[X_{v,i}^{(2)} = 1 \right] \cdot \mathbf{Pr} \left[\forall u \in \Gamma_\pi(v), S_u^{(2)} = \emptyset \right] \geq \frac{x_{v,i}^{(1)}}{8\rho} , \end{aligned}$$

which proves the Proposition for Case 2. □

Finally, to prove the theorem we note that by splitting the solution into two parts and returning the better output, we lose only a factor of 2 in the approximation guarantee. For the expected social welfare it holds $\max_{l \in \{1,2\}} \sum_{v \in V} \sum_{i=1}^k b_v(i) \cdot x_{v,i}^{(l)} \geq b^*/2$. □

3.2 Edge-Weighted Conflict Graphs

Allocating the channels is much more involved in the case of edge-weighted conflict graphs due to the asymmetry of interference constraints. In the unweighted case the simple greedy allocation only has to make sure there are no edges to vertices on the same channel. This is unsuitable now since adding a user might violate constraints at previously added users – even though constraints are satisfied for the currently added user.

Having obtained the $d_v^{(l)}$ values in the described way, we first consider only the incoming weight from users of smaller index like in the unweighted case. If the incoming weight from previous users is too high, i.e., $\sum_{u \in V, \pi(u) < \pi(v)} d_u^{(l)} \cdot \bar{w}(u, v) \geq k/32$, we remove all channels from the user and set $d_v^{(l)} := 0$. However, unlike in the unweighted case, this does not yet guarantee the existence of an allocation. The crucial difference occurs in the last step, where the allocation is derived. This step is performed differently for the two solutions of the decomposition. For the case in which each user was assigned at most $k/8$ channels, the allocation is made in a randomized fashion in Algorithm ALLOCATE(1). For the other case, the allocation is made deterministically in Algorithm ALLOCATE(2). Unlike in the unweighted case, in both cases the resulting allocation will not include all users at a time but only allocate channels to a subset of the originally chosen users.

Algorithm 2: LP-Rounding for Symmetric Valuations and Weighted Conflict Graphs

- 1 Decompose an optimal solution x to LP (1) into two solutions $x^{(1)}$ and $x^{(2)}$ as follows:
Set $x_{v,i}^{(1)} = x_{v,i}$ if $i \leq k/8$ and $x_{v,i}^{(1)} = 0$ otherwise; set $x^{(2)} = x - x^{(1)}$.
 - 2 **for** $l \in \{1, 2\}$ **do**
 - 3 **for** $v \in V$ *in increasing order of π values* **do**
 - 4 With probability $\frac{x_{v,i}^{(l)}}{64\rho}$ set $d_v^{(l)} := i$
 - 5 Set $d_v^{(l)} := 0$ if $\sum_{u \in V, \pi(u) < \pi(v)} d_u^{(l)} \cdot \bar{w}(u, v) \geq k/32$
 - 6 Run Algorithm ALLOCATE(l) on $d^{(l)}$, let $S^{(l)}$ be the result
 - 7 Return the better one of the solutions $S^{(1)}$ and $S^{(2)}$
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Theorem 3. *Algorithm 2 returns a feasible allocation of social welfare at least $\Omega(b^*/\rho \cdot (\log n + \log k))$ in expectation.*

In order to show the bound, we will show that both LP solutions are rounded to feasible allocations that are in expectation at most a $O(\rho \cdot (\log n + \log k))$ factor worse than the respective LP solution.

As a first step, we analyze the input given in terms of the number of channels for each user. In particular, we show that an allocation satisfying all of these demands simultaneously would in expectation be at most a $1/128\rho$ factor worse than the fractional solution.

Proposition 4. *For $l \in \{1, 2\}$ and the expected social welfare of $d^{(l)}$ we have*

$$\mathbf{E} \left[\sum_{v \in V} b_v(d_v^{(l)}) \right] \geq \frac{1}{128\rho} \cdot \sum_{v \in V} \sum_{i=1}^k b_v(i) \cdot x_{v,i}^{(l)}.$$

Proof. For all $v \in V$, $i \in [k]$, $l \in \{0, 1\}$ let $X_{v,i}^{(l)}$ be a 0/1 random variable indicating if in the rounding stage $d_v^{(l)}$ is set to i . We know that $\Pr[X_{v,i}^{(l)} = 1] = \frac{x_{v,i}^{(l)}}{4\rho}$. Let $Y_{v,i}^{(l)}$ be the respective 0/1 random variable at the time when the allocation algorithm is started.

We have to bound $\Pr[Y_{v,i}^{(l)} = 0 \mid X_{v,i}^{(l)} = 1]$. This is the probability that the weight bound in line 5 is exceeded. By Markov inequality, we get

$$\Pr[Y_{v,i}^{(l)} = 0 \mid X_{v,i}^{(l)} = 1] \leq \frac{32}{k} \cdot \mathbf{E} \left[\sum_{u \in V, \pi(u) < \pi(v)} d_u^{(l)} \bar{w}(u, v) X_{v,i}^{(l)} \right].$$

Applying linearity of expectation and the fact we have an LP solution this is

$$\frac{32}{k} \cdot \sum_{\substack{u \in V \\ \pi(u) < \pi(v)}} d_u^{(l)} \cdot \bar{w}(u, v) \cdot \frac{x_{v,i}^{(l)}}{64\rho} \leq \frac{1}{2}.$$

In total, we obtain

$$\mathbf{E} \left[\sum_{v \in V} b_v(d_v^{(l)}) \right] = \sum_{v \in V} \sum_{i=1}^k b_v(i) \cdot \Pr[Y_{v,i}^{(l)} = 1] \geq \frac{1}{128\rho} \sum_{v \in V} \sum_{i=1}^k b_v(i) \cdot x_{v,i}^{(l)}.$$

□

In the two following subsections, we consider the two allocation algorithms and show that in either case a feasible allocation of social welfare at least $\Omega(\sum_{v \in V} b_v(d_v^{(l)}) / (\log n + \log k))$ is computed.

3.2.1 ALLOCATE(1): Allocation algorithm for “small” sets

From a preliminary selection of numbers of channels Algorithm ALLOCATE(1) generates a feasible allocation in which $d_v \leq k/8$ for each $v \in V$ and $\sum_{u \in V, \pi(u) < \pi(v)} d_u \bar{w}(u, v) < k/32$. The idea is that a number of allocations are computed having the property that each user is considered in exactly one of these allocations. Each allocation is computed by first selecting a subset of all users and then performing k randomized contention resolution steps. We iterate over the k channels, and for each channel we let each user v independently perform a random experiment. With probability $8d_v/k$ it receives this channel tentatively. If the user received d_v channel it keeps the respective channels in this allocation is dropped from consideration. All other users are allocated in later rounds. The main argument to show that this yields feasibility and provides the desired bound on the approximation factor relies on a suitable tracking of the degrees during the contention resolution process.

Lemma 5. *The allocation has social welfare at least $\Omega(\sum_{v \in V} b_v(d_v) / (\log n + \log k))$ with high probability, i.e., with probability at least $1 - (nk)^{-c}$ for any constant $c > 1$.*

Proof. In order to show this bound, it suffices to prove that

$$\mathbf{E} \left[\sum_{v \in V_{t+1}} d_v \mid V_t \right] \leq \frac{3}{4} \sum_{v \in V_t} d_v.$$

Algorithm ALLOCATE(1): Channel allocation for users that require at most $k/8$ channels.

```

1 Set  $V_0 := V$  and  $t := 0$ ;
2 while  $V_t \neq \emptyset$  do
3   for  $u \in V_t$  in decreasing order of  $\pi$  values do
4     if  $\sum_{v \in H_t} d_v \cdot \bar{w}(u, v) < k/32$  then
5       add  $u$  to  $H_t$  and for each  $j \in [k]$  set  $X_{v,j}$  independently to 1 with probability
6          $8d_v/k$ ;
7   for  $v \in H_t$  do
8     For each  $j \in [k]$  set  $Y_{v,j} = 1$  if  $\sum_{u \neq v} \bar{w}(u, v) \cdot X_{u,j} < 1$ ;
9     if  $\sum_{j \in [k]} Y_{v,j} \geq d_v$  then
10       set  $S_v^t$  to an arbitrary subset of  $d_v$  channels  $j$  with  $Y_{v,j} = 1$ ;
11 Let  $V_{t+1}$  be the set of users who have not been allocated anything and set  $t := t + 1$ ;
12 Return the best one of the allocations  $S^1, S^2, S^3, \dots$ ;

```

Using Markov inequality this implies that for each constant $c > 1$ the probability that the set V_t with $t = (c + 1) \log(nk) / \log(4/3)$ is not empty is at most

$$\Pr \left[\sum_{v \in V_t} d_v \geq 1 \right] \leq \mathbf{E} \left[\sum_{v \in V_t} d_v \right] \leq \left(\frac{3}{4} \right)^t nk = (nk)^{-c}.$$

Thus with high probability at most $O(\log(\sum_{v \in V} d_v)) = O(\log n + \log k)$ allocations are computed.

We prove the bound in two steps. First, we show that the sum of demands in the set H_t is at least half of the total demands in V_t . Afterwards, we observe that for a user in H_t , the probability to be included is at least $\frac{1}{2}$.

Claim 6.

$$\sum_{v \in H_t} d_v > \frac{1}{2} \sum_{v \in V_t} d_v.$$

Proof. Each user $u \in V_t \setminus H_t$ was excluded from H_t because we have

$$\sum_{\substack{v \in H_t \\ \pi(u) < \pi(v)}} d_v \cdot \bar{w}(u, v) \geq \frac{k}{32}.$$

Taking the sum, weighted by the respective d_u value, we get

$$\sum_{u \in V_t \setminus H_t} d_u \cdot \sum_{\substack{v \in H_t \\ \pi(u) < \pi(v)}} d_v \cdot \bar{w}(u, v) \geq \sum_{u \in V_t \setminus H_t} d_u \cdot \frac{k}{32}.$$

On the other hand, we have

$$\sum_{u \in V_t \setminus H_t} d_u \cdot \sum_{\substack{v \in H_t \\ \pi(u) < \pi(v)}} d_v \cdot \bar{w}(u, v) = \sum_{v \in H_t} d_v \cdot \sum_{\substack{u \in V_t \setminus H_t \\ \pi(u) < \pi(v)}} d_u \cdot \bar{w}(u, v) < \sum_{v \in H_t} d_v \cdot \frac{k}{32}.$$

Assembling the two bounds yields the claim. \square

Claim 7. *The probability for each user $v \in H_t$ to be included in the allocation is at least $\frac{1}{2}$.*

Proof. A user $v \in V$ is not included in the allocation if there is a set $M \subseteq [k]$ with $|M| \geq k - d_v$ such that $Y_{v,j} = 0$ for all $j \in M$.

Let us first consider a single channel j . In order to have $Y_{v,j} = 1$, two independent events have to occur: First, we have to have $X_{v,j} = 1$ and second $\sum_{u \neq v} \bar{w}(u, v) \cdot X_{u,j} < 1$. The probability for the first one is defined in the algorithm, the second one can be bounded by the Markov inequality to get

$$\Pr[Y_{v,j} = 1] \geq \Pr[X_{v,j} = 1] \cdot \left(1 - \mathbf{E} \left[\sum_{u \neq v} \bar{w}(u, v) \cdot X_{u,j} \right]\right) = \frac{8d_v}{k} \cdot \left(1 - \sum_{u \neq v} \bar{w}(u, v) \cdot \frac{8d_u}{k}\right) \geq \frac{4d_v}{k}.$$

Now consider a block B of $\lfloor \frac{k}{2d_v} \rfloor \geq \frac{3k}{8d_v}$ consecutive channels. Since the random experiments are independent, for such a block B the probability of $\sum_{j \in B} Y_{v,j} = 0$ is at most

$$\left(1 - \frac{4d_v}{k}\right)^{\frac{3k}{8d_v}} \leq \exp\left(-\frac{3}{2}\right) \leq \frac{1}{4}.$$

Since there are k channels in total, we have at least $2d_v$ blocks in total. For each of these blocks, the probability of v getting no channel in this block is at most $\frac{1}{4}$. This is, the expected number of blocks B in which $\sum_{j \in B} Y_{v,j} = 0$ is at most $\frac{d_v}{2}$. Using the Markov inequality, the probability that there are more than d_v blocks without a channel for v is less than $\frac{1}{2}$. Thus, with probability at least $\frac{1}{2}$, v gets at least 1 channel in at least d_v blocks. This yields the claim. \square

Combining these two insights, we get the desired bound which proves the lemma. \square

3.2.2 ALLOCATE(2): Allocation algorithm for “large” sets

The allocation for the case that $d_v \geq k/8$ or $d_v = 0$ for all $v \in V$ is performed by Algorithm ALLOCATE(2). Here, we iterate starting with $t = 1$. In each iteration, a subset H_t of all users is selected by going through the remaining users in decreasing order of π . If for a user v we have $\sum_{v \in H_t} d_v \cdot \bar{w}(u, v) < k/32$, it is added to H_t . However, in this case the allocation is immediately carried out in a direct way: Each user that is added to H_t is allocated an arbitrary set of d_v channels, e.g. the first ones. This iteration is repeated with the remaining users that did not get allocated anything until every user $v \in V$ has been allocated d_v channels in one iteration t . Finally, the algorithm picks the best of the allocations computed in any single iteration.

Proposition 8. *The algorithm computes at most $O(\log n + \log k)$ allocations and all of them are feasible.*

Proof. Using exactly the same arguments as in Claim 6 above, we observe

$$\sum_{v \in H_t} d_v > \sum_{u \in V_t \setminus H_t} d_u,$$

which shows that at most $O(\log n + \log k)$ allocations are computed.

The allocations are feasible since the sum of incoming weights on any channel is bounded by

$$\sum_{u \in H_t} \bar{w}(u, v) \leq \frac{8}{k} \sum_{u \in H_t} d_u \cdot \bar{w}(u, v) < \frac{8}{k} \cdot \frac{k}{32} = \frac{1}{4}.$$

\square

Algorithm ALLOCATE(2): Channel allocation for users that require at least $k/8$ channels.

```

1 Set  $V_0 := V$  and  $t := 0$ ;
2 while  $V_t \neq \emptyset$  do
3   for  $u \in V_t$  in decreasing order of  $\pi$  values do
4     if  $\sum_{v \in H_t} d_v \cdot \bar{w}(u, v) < k/32$  then
5       └ Add  $u$  to  $H_t$  and set  $S_v^t = \{1, \dots, d_v\}$ ;
6   └ Let  $V_{t+1}$  be the set of users who have not been allocated anything and set  $t := t + 1$ ;
7 Return the best one of the allocations  $S^1, S^2, S^3, \dots$ ;

```

3.3 Truthfulness

To turn the approximation algorithms from the previous section into truthful mechanisms, we follow the idea by Lavi and Swamy [24] using the randomized meta-rounding technique [5] to obtain a MIDR mechanism. Our approach is similar to the one for general secondary spectrum auctions [20]. Linear program (1) are standard packing LPs that allow to set up a separation LP to decompose an optimal fractional solution scaled down by some approximation factor larger than the integrality gap. Via this new LP we derive a decomposition into integral solutions that represent feasible solutions for the channel allocation problem. An optimal solution to the decomposition LP is a probability distribution, by which we can randomly pick a feasible solution in the resulting mechanism. As usual, charging scaled VCG payments results in a mechanism that is truthful in expectation.

The decomposition LP uses exponentially many variables (probabilities for every possible feasible solution) but only polynomially many constraints (decomposition of each non-zero variable in the fractional optimum of LP (1)). Thus, the dual of the LP can be solved using the ellipsoid method with a suitable separation oracle. The latter can be constructed from our algorithms presented in the last section, as they verify the correct approximation factor used for scaling in the decomposition LP. At this point it is important to remark that the algorithms were defined to be randomized. Therefore, the running time of the ellipsoid method would only be polynomial in expectation. However, all of our algorithms can be derandomized using standard techniques. The randomization in Algorithms 1 and 2 only depends on pairwise independence. Algorithm ALLOCATE(1) can be made deterministic by using a combination of pairwise-independence and conditional-expectation techniques. Under these conditions, the desired decomposition can be found in polynomial time.

A main drawback of this method is that the dual variables of the decomposition must be interpreted as valuations of a new channel allocation problem. Here assumptions like symmetry or submodularity cannot be made, and algorithms for such special classes of valuations might not be applicable. However, in our case the symmetry assumption is encoded directly into LPs (1) by setting up variables for each *number* and each not *set* of channels. This property carries over to the decomposition dual and our algorithms can be applied.

4 Matroid-Rank-Sum Valuations

In this section, we treat the class of so-called *matroid rank sum (MRS) valuations*, in which b_v for each bidder is a weighted sum of matroid rank functions. This covers all frequently considered

submodular valuation functions such as, e.g., coverage functions, matroid weighted-rank functions, and any convex combinations of these.

For ordinary combinatorial auctions, Dughmi et al. [12] present an MIDR mechanism. The range is given by all solutions to a linear relaxation of the item-allocation problem. Rounding is done via a non-standard randomized rounding scheme called *Poisson rounding* in [12]. Finding the optimal distribution implies finding the fractional allocation that will achieve best social welfare in expectation in the rounding stage. The Poisson scheme is a *convex rounding scheme*, for which finding the best fractional allocation becomes a convex program with objective function being expected social welfare.

Unfortunately, the Poisson rounding scheme is tailored to fit to ordinary combinatorial auctions. The rounding is performed item-wise – when $x_{i,j}$ is the fractional allocation of item j to bidder i , then j is fully given to i with probability $1 - e^{-x_{i,j}}$. With the remaining probability no bidder receives j . Unlike items, the channels in our case can be given to multiple users, and it takes significantly more effort to build a convex rounding scheme. In the following we present our approach for this case. We follow the conventions in [12], in particular, with respect to representation of MRS valuations using lottery-value oracles. In particular, we will show the following theorem.

Theorem 9. *There is a truthful mechanism for MRS valuations that runs in expected polynomial time and returns a feasible allocation representing a $O(\rho)$ -approximation for unweighted and a $O(\rho \cdot \log n)$ -approximation for edge-weighted conflict graphs.*

4.1 Defining the Range

We define the distributional range \mathcal{D} in this section and discuss why it is sufficiently large to get good approximations. Our starting point are all fractional solutions x fulfilling the following linear constraints:

$$\sum_{\substack{u \in V \\ \pi(u) < \pi(v)}} \bar{w}(u, v) \cdot x_{u,j} \leq \rho \quad \text{for all } v \in V, j \in [k] \quad (2a)$$

$$0 \leq x_{v,j} \leq 1 \quad \text{for all } v \in V, j \in [k] \quad (2b)$$

Algorithm 3: Rounding scheme for a given solution x .

```

1 for  $j \in [k]$  do
2   Draw  $p_j$  uniformly for  $[0, 1]$ ;
3   Decompose  $(x_{v,j})_{v \in V}$  such that  $x = \frac{1}{\alpha} \sum_l \lambda_l g_l$  and  $\sum_l \lambda_l = 1$ ;
4   Let  $l'$  be the minimal  $l$  for which  $\sum_{l < l'} \lambda_l < p_j$ ;
5   Allocate  $g_{l'}$  tentitatively;
6   Remove each  $v \in V$  from solution with a further probability of  $p_{v,j} = \frac{1 - e^{-x_{v,j}/(2\alpha)}}{\frac{x_{v,j}}{\alpha}}$ ;
```

For each channel we pick a feasible independent set separately in our rounding scheme Algorithm 3. For each channel j the corresponding fractional solution $x_{\cdot,j}$ is decomposed into polynomially many independent sets using parameter α discussed below. The algorithm selects one of these at random. The decomposition can be computed in polynomial time using randomized meta-rounding [5, 24] in combination with an appropriate rounding scheme. Afterwards, each user v is

removed from the solution by an independent random experiment rendering the total probability for v to receive channel j to be exactly $1 - e^{-x_{v,j}/2\alpha}$. Note that $p_{v,j}$ must be a valid probability with $p_{v,j} \in [0, 1]$. Here we observe that since numerator and denominator are both positive, $p_{v,j}$ also is. $p_{v,j} \leq 1$ because $1 - e^{x_{v,j}/(2\alpha)} \leq \frac{x_{v,j}}{2\alpha}$, for any $\alpha \geq 1$. Consequently, the range \mathcal{D} is given by all probability distributions resulting from our rounding scheme applied to fractional solutions of (2a) and (2b).

We have to specify the parameter α , which ensures that the decomposition of $x_{\cdot,j}$ exists. We interpret $x_{\cdot,j}$ as solution to a linear program to maximize $\sum_{v \in V} a_v \cdot x_{v,j}$ subject to the constraints (2a) and (2b) for channel j . This is essentially a linear relaxation for a single channel allocation problem with some valuations a_v . We denote by α the integrality gap of this program with respect to *feasible* independent sets (Note that the constraints (2a) allow integer solutions x that represent infeasible independent sets). For this program we can verify an integrality gap of $\alpha = O(\rho \cdot \log n)$ for feasible independent sets using, e.g., the LP-rounding algorithm for edge-weighted conflict graphs from [20]. For unweighted conflict graphs, the simpler LP-rounding algorithm from [20] yields $\alpha = O(\rho)$. Here, the simple greedy algorithm of [1] (for details see Section 5 below) can even be shown to yield $\alpha = \rho$.

For application of the randomized metarounding framework, we need an algorithm verifying an integrality gap α . This allows to construct a decomposition LP and its dual, which can be solved in polynomial time using the ellipsoid method, where the algorithm acts as separation oracle (for details on this method see [5, 24]). Note that α can merely be seen as a parameter that serves to scale a fractional solution x into a region where a decomposition into (feasible) integral solutions exists – independent of any objective function. The reason we interpret it as integrality gap of an optimization problem is that the dual of the decomposition LP allows an approximation algorithm verifying the gap to be used to separate the dual and derive the required decomposition in polynomial time. The reason we do not simply radically overestimate α is that it does play a central role when we discuss the approximation factor of our rounding scheme.

For a given distribution, the expected social welfare of the returned allocation is exactly

$$\sum_{v \in V} \sum_{T \subseteq [k]} b_v(T) \prod_{j \in T} (1 - e^{-x_{v,j}/(2\alpha)}) \prod_{j \notin T} e^{-x_{v,j}/(2\alpha)} . \quad (3)$$

For the case of MRS functions, this function is concave, as we will observe in more detail below. Therefore, the best distribution in the range can be arbitrarily approximated by solving a convex program, maximizing the concave objective (3) subject to linear constraints (2a) and (2b).

As previously mentioned, the size of the range affects approximation factor and tractability. Concerning the approximation factor, we can show that the social welfare of the optimal allocation is at most an $O(\alpha)$ -factor above the expected social welfare of the best distribution in the range.

Lemma 10. *The optimal distribution within the range is $O(\alpha)$ -approximate in expectation when valuations are submodular. Hence, in expectation, the solution of our rounding scheme is a $O(\rho)$ -approximation for unweighted and a $O(\rho \cdot \log n)$ -approximation for edge-weighted conflict graphs.*

Proof. The optimal allocation S^* corresponds to a feasible solution x^* of the convex program. However, x^* is not always rounded to S^* but also to worse allocations. We bound the expected welfare of the received allocation in terms of that of S^* . This then yields the upper bound on the approximation ratio. The probability of each user v of being allocated channel j in rounding is exactly $1 - e^{-x_{v,j}^*/(2\alpha)}$. We denote $b(S^*) = \sum_{v \in V} b_v(S^*(v))$ and use Proposition C.4 in [12].

This yields an expected social welfare of the rounded allocation of at least $(1 - e^{-1/(2\alpha)}) \cdot b(S^*) \geq (1 - e^{-1}) \cdot (2\alpha)^{-1} \cdot b(S^*)$ due to concavity. Thus, the result of rounding the best distribution is at most a factor of $O(\alpha)$ worse. \square

4.2 Sampling the MIDR Distribution

The expected social welfare when rounding a fractional solution x is given by (3). Fortunately, this function is concave in terms of x meaning an optimal fractional solution can be approximated arbitrarily well in polynomial time. However, to make the mechanism truthful in expectation, we are, in principle, required to solve the given convex program *exactly*.

Since this is not possible, Algorithm 4 devises a way to simulate an exact solution in expected polynomial time. It returns an allocation in which each bidder has exactly the same probability as in Algorithm 3 to get a channel. It requires us to compute δ -estimates – a solution x of the convex program such that $x_{v,j}^* - \delta \leq x_{v,j} \leq x_{v,j}^* + \delta$ for all v, j . To simplify the presentation, we assume that this can be computed in time $\text{poly}(n, k, \log(1/\delta))$. For details on this issue, see Section 4.3.

Algorithm 4: Simulating Algorithm 3 with estimates of the optimal convex-program solution.

```

1 for  $j \in [k]$  do
2   Draw  $p_j$  uniformly from  $[0, 1)$  and let  $r$  be the minimal  $t$  for which  $p_j \geq 1 - 2^{-t+1}$  ;
3   Set  $x^0 = 0$ ;
4   for  $t = 1, \dots, r$  do
5     Compute  $\delta^t$ -estimate  $x^t$ , where  $\delta^t = 1/(n \cdot 2^{t+1})$ ;
6     Let  $y_v^t = \max\{y_v^{t-1}, x_{v,j}^t - \delta^t\}$ ;
7     Decompose  $y^r - y^{r-1}$  such that  $y^r = \frac{1}{2\alpha} \sum_l \lambda^{r,l} g^{r,l}$  with  $\sum_l \lambda^{r,l} = 2^{-r}$ ;
8     Let  $l'$  be the minimum  $l$  such that  $p_j > 1 - 2^{-r-1} + \sum_{l < l'} \lambda_l$  ;
9     Tentatively allocate  $g^{r,l'}$  ;
10    Remove each  $v \in V$  from solution with further probability  $p_{v,j} = \frac{2\alpha(e^{-y_v^{t-1}/(2\alpha)} - e^{-y_v^t/(2\alpha)})}{y_v^t - y_v^{t-1}}$ 

```

Proposition 11. *The desired decomposition $(g^{r,l}, \lambda^{r,l})_l$ exists and can be computed in polynomial time.*

Proof. We distinguish between the two cases $r = 1$ and $r \geq 2$.

In the case of $r = 1$, y^r fulfills equations (2a) and (2b). Here we can apply the decomposition as described above. Using the algorithms from [20] verifying integrality gaps of $\alpha = O(\rho)$ or $\alpha = O(\rho \cdot \log n)$, we can solve the decomposition LP of the meta-rounding framework and decompose $y^b = \frac{1}{\alpha} \tilde{\lambda}^{r,l} g^{r,l}$ with $\sum_l \tilde{\lambda}^{r,l} = 1$ where $g^{r,l}$ are integral solutions corresponding to independent sets. The running time is polynomial in n and k . Setting $\lambda^{r,l} = \frac{1}{2} \tilde{\lambda}^{r,l}$ for all l yields the desired composition.

For the case $r \geq 2$, we use the fact that x^{r-1} is already a $1/(n2^r)$ -estimate. This yields that $0 \leq y_v^r - y_v^{r-1} \leq 1/(n2^{r-1})$. Therefore, it is possible to decompose $y^r - y^{r-1}$ to the trivial single-vertex independent sets. Formally, we consider an arbitrary ordering of the users v_1, \dots, v_n , e.g. the one given by π . We set $g_{v_i}^{r,l} = 1$ and $g_v^{r,l} = 0$ if $v_i \neq v$. The weights are set to $\lambda^{r,l} = \frac{1}{2\alpha}(y_{v_i}^r - y_{v_i}^{r-1})$. This yields that $\sum_{l=1}^n \lambda^{r,l} \leq \sum_{l=1}^n \frac{1}{2\alpha} \cdot \frac{1}{n2^{r-1}} \leq 2^{-r}$. The remaining weight is assigned to the all-zero fractional solution. \square

Proposition 12. *For the probability of being removed we have $p_{v,j} \in [0, 1]$.*

Proof. Since $y_v^{t-1} \leq y_v^t$ for all $v \in V$, the probability is at least 0. Furthermore, we have

$$\begin{aligned} \frac{2\alpha \left(e^{-y_v^{t-1}/(2\alpha)} - e^{-y_v^t/(2\alpha)} \right)}{y_v^t - y_v^{t-1}} &= \frac{2\alpha e^{-y_v^{t-1}/(2\alpha)} \left(1 - e^{-(y_v^t - y_v^{t-1})/(2\alpha)} \right)}{y_v^t - y_v^{t-1}} \\ &\leq \frac{2\alpha \left(1 - e^{-(y_v^t - y_v^{t-1})/(2\alpha)} \right)}{y_v^t - y_v^{t-1}} \\ &\leq 1 . \end{aligned}$$

□

Proposition 13. *For each user $v \in V$ and each channel $j \in [k]$ the probability to receive j is exactly $1 - e^{-x_{v,j}^*/(2\alpha)}$.*

Proof. Let r be defined as in the algorithm. Let us first consider the conditional probability of getting the channel given that $r = t$ for some t .

$$\begin{aligned} \Pr[v \text{ gets } j \mid r = t] &= \Pr \left[g_v^{r,l'} = 1 \mid r = t \right] \cdot \frac{2\alpha \left(e^{-y_v^{t-1}/(2\alpha)} - e^{-y_v^t/(2\alpha)} \right)}{y_v^t - y_v^{t-1}} \\ &= \frac{2^t (y_v^{t-1} - y_v^t)}{2\alpha} \cdot \frac{2\alpha \left(e^{-y_v^{t-1}/(2\alpha)} - e^{-y_v^t/(2\alpha)} \right)}{y_v^t - y_v^{t-1}} \\ &= 2^t \left(e^{-y_v^{t-1}/(2\alpha)} - e^{-y_v^t/(2\alpha)} \right) \end{aligned}$$

We get

$$\Pr[v \text{ gets } j] = \sum_{t=1}^{\infty} \Pr[r = t] \cdot \Pr[v \text{ gets } j \mid r = t] = \sum_{t=1}^{\infty} 2^{-t} \cdot 2^t \left(e^{-y_v^{t-1}/(2\alpha)} - e^{-y_v^t/(2\alpha)} \right) = 1 - e^{-x_{v,j}^*} ,$$

where the last step is due to the fact that y_v^t converges to $x_{v,j}^*$ as $t \rightarrow \infty$. □

Proposition 14. *Assuming that the δ -estimates can be computed in time $\text{poly}(n, k, \log(1/\delta))$, the expected running time of Algorithm 4 is polynomial in n and k .*

Proof. Let us first consider the running time for the case that $r = t$ for some fixed t . If this case the δ -estimates in lines 5–7 can be computed in time $\sum_{i=1}^t \text{poly}(n, k, \log(2^{i+1}n)) = \text{poly}(n, k, t)$. The remaining computations take time $\text{poly}(n, k)$. As a consequence, the expected running time of the algorithm is $\sum_{t=1}^{\infty} \Pr[r = t] \cdot \text{poly}(n, k, t) = \sum_{t=1}^{\infty} 2^{-t} \cdot \text{poly}(n, k, t) = \text{poly}(n, k)$, where the last step is due to a geometric series. □

4.3 Computing δ -Estimates

Algorithm 4 only runs in expected polynomial time when assuming that a δ -estimate of the convex program can be computed in time $\text{poly}(n, k, \log(1/\delta))$. The reasoning why we assume this is

essentially the same as in [12]. However, for the sake of completeness, we present the most important steps in this section.

First of all, we have to observe that the objective function is concave when all player valuations are MRS.

Lemma 15. *Our rounding scheme is convex when player valuations are MRS.*

Proof. Due to $\mathbf{E} [\sum_{v \in V} b_v(S_v)] = \sum_{v \in V} \mathbf{E} [b_v(S_v)]$, the result follows when $\mathbf{E} [b_v(S_v)]$ is concave for all v . By construction the probability for each user to be allocated channel j is exactly $1 - e^{-x_{v,j}/(2\alpha)}$. Therefore each $\mathbf{E} [b_v(S_v)]$ can be written as

$$\sum_{T \subseteq [k]} b_v(T) \prod_{j \in T} (1 - e^{-x_{v,j}/(2\alpha)}) \prod_{j \notin T} e^{-x_{v,j}/(2\alpha)} .$$

We only have to prove that this function is concave over $(0, 1)^k$.

Dughmi et al. [12] show that the function $G: \mathbb{R}^k \rightarrow \mathcal{R}$ with

$$G(x_1, \dots, x_k) = \sum_{T \subseteq [k]} b(T) \prod_{j \in T} (1 - e^{-x_j}) \prod_{j \notin T} e^{-x_j}$$

is concave over $x \in (0, 1)^k$ when b is MRS.

For $\mathbf{E} [b_v(S_v)] = G(x/(2\alpha))$ this also yields concavity since for any $\xi \in [0, 1]$

$$G\left(\frac{\xi x + (1 - \xi)y}{(2\alpha)}\right) = G\left(\xi \frac{x}{(2\alpha)} + (1 - \xi) \frac{y}{(2\alpha)}\right) \geq \xi G\left(\frac{x}{(2\alpha)}\right) + (1 - \xi) G\left(\frac{y}{(2\alpha)}\right) .$$

□

This immediately yields the following claim when taking into consideration that the constraints are linear.

Claim 16. *There is an algorithm in the lottery-value oracle model that, given an instance of spectrum auctions with edge-weighted conflict graphs on n bidders and k channels and an approximation parameter $\epsilon > 0$, runs in $\text{poly}(n, k, \log(1/\epsilon))$ time and returns a $(1 - \epsilon)$ -approximate solution to the convex program.*

This yields the following result for δ -estimates. Suppose we are in the well-conditioned case, i.e., on any line in the feasible set the second derivative of the objective function is at least $\lambda = \frac{\sum_{v \in V} b_v([k])}{2^{\text{poly}(n, k)}}$. Then a δ -estimate can be computed by computing an $(1 - \epsilon)$ -approximate solution of the convex program with $\epsilon = \frac{\delta^2}{2 \sum_{v \in V} b_v([k])}$. This solution can be computed in time $\text{poly}(n, k, \log(1/\delta))$.

4.3.1 Guaranteeing Good Conditioning

In general, the bound on the second derivative does not necessarily have to hold. Therefore, the algorithm is modified as given in Algorithm 5.

After having run Algorithm 4, the resulting allocation is discarded with probability $\mu = 2^{-nk}$. Instead a trivial allocation is returned, in which either only a single user gets allocated all channels or even no channels are allocated at all, as determined by another random experiment. However,

Algorithm 5: Modified MIDR Algorithm.

```
1 Run Algorithm 4, let  $S$  be the resulting allocation;
2 Let  $\beta$  be  $\frac{1}{nk} \sum_{v \in V} |S(v)|$ ;
3 Draw  $q_1$  uniformly at random from  $[0, 1]$ ;
4 if  $q_1 \leq \mu$  then
5   Set  $S(v) := \emptyset$  for all  $v \in V$ ;
6   Draw  $q_2$  uniformly at random from  $[0, 1]$ ;
7   if  $q_2 \leq \beta$  then
8     Choose some user  $v^* \in V$  uniformly at random;
9     Set  $S(v^*) = [k]$  and  $S(v) = \emptyset$  for all  $v \neq v^*$ ;
```

since this action is only taken with probability $1 - \mu = 1 - o(1)$, the approximation factor is not affected.

On the contrary, we can show that the expected social welfare changes, now having a curvature of at least λ . This is the missing piece to build the δ -estimates necessary to run the algorithm.

In order to determine the precise expected social welfare of the modified algorithm, we have to first quantify the probability that the initially computed solution is discarded. This is done with probability β , which depends on the previous outcome. For the expectation, we know

$$\mathbf{E}[\beta] = \mathbf{E}\left[\frac{1}{nk} \sum_{v \in V} |S(v)|\right] = \frac{1}{nk} \sum_{j=1}^k \sum_{v \in V} \Pr[j \in S(v)] = \frac{1}{nk} \sum_{j=1}^k \sum_{v \in V} \left(1 - e^{x_{v,j}/(2\alpha)}\right) .$$

Therefore the expected social welfare is

$$\begin{aligned} & (1 - \mu) \cdot \mathbf{E}[b(S)] + \mu \cdot \mathbf{E}[\beta] \frac{1}{n} \sum_{v \in V} b_v([k]) \\ &= (1 - \mu) \cdot \mathbf{E}[b(S)] + \frac{\mu}{n^2 k} \cdot \left(\sum_{j=1}^k \sum_{v \in V} \left(1 - e^{x_{v,j}/(2\alpha)}\right) \right) \cdot \left(\sum_{v \in V} b_v([k]) \right) . \end{aligned}$$

Since both parts of the outer sum are non-negative, it suffices to bound the curvature of the second one. The curvature of $\left(\sum_{j=1}^k \sum_{v \in V} \left(1 - e^{x_{v,j}/(2\alpha)}\right)\right)$ is at least $(e(2\alpha)^2)^{-1}$. Therefore, the curvature of the second part is at least

$$\frac{\mu}{n^2 k} \cdot \frac{1}{e(2\alpha)^2} \cdot \left(\sum_{v \in V} b_v([k]) \right) = \lambda .$$

As a consequence, the modified algorithm can be run with δ -approximates as described above with a resulting running time that is $\text{poly}(n, k)$ in expectation.

5 Discussion and Open Problems

While the mechanisms presented in previous sections obtain near-optimal guarantees on social welfare, they have some drawbacks for application in practice. A serious problem are running times

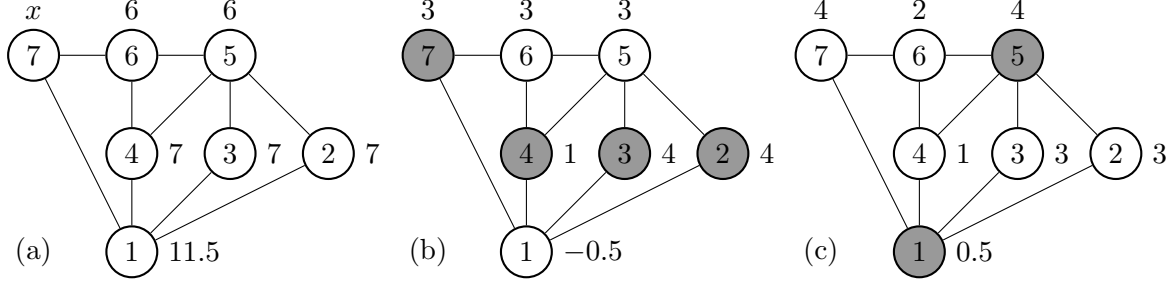


Figure 1: Example for non-monotonicity of the greedy algorithm. In part (a), the number inside the circle denotes the vertex’s index in the π -ordering, the one outside its reported valuation. If bidder 7 reports $x = 3$, he is included in the solution; if he bids up to $x = 4$, he is dropped by the algorithm. Parts (b) and (c) depict the resulting values and the independent sets at the end of the execution of the algorithm for each case, respectively.

– for MRS valuations our mechanism obtains polynomial running time only in expectation. For symmetric valuations, we obtain polynomial worst-case running times, but the convex optimization techniques needed to apply randomized meta-rounding often have prohibitive running times for large practical problem instances. Thus, let us briefly discuss designing fast and simple mechanisms. How can we design a good and simple deterministic mechanism to incentivize truth-telling among bidders?

To our knowledge, there are only two algorithmic approaches to the channel assignment problem that yield approximation guarantees in the order of $O(\rho)$. One approach is rounding of suitably relaxed packing LPs, which turned out to be very successful in this and our previous work [20]. While pairwise independence can be used to make these algorithms deterministic, they require randomization to guarantee truthfulness and fail for deterministic truthfulness. The other approach was proposed for the simplest case of a single channel and unweighted conflict graphs, i.e., the maximum weighted independent set problem. It is a simple greedy algorithm due to Akcoglu et al [1] which first considers vertices one by one in reverse of the ordering of π . If vertex v is under consideration, its current value is subtracted from the value of each backward neighbor. If the value of a vertex drops to 0 or below before it is under consideration in the ordering, this vertex is removed. Finally, the algorithm makes a second pass over the surviving vertices, this time in forward ordering of π , and greedily adds each vertex to the independent set if possible. It can be shown using a local ratio argument that it provides a ρ -approximation [34].

It is tempting to believe that this algorithm is monotone and delivers a deterministically truthful mechanism. Unfortunately, this is not the case, see our example in Figure 1. The problem is that the algorithm makes a second pass over the vertices which introduces non-trivial dependencies among bids and acceptance decisions. Nevertheless, we show how to turn it into a monotone algorithm by spending a $\log n$ factor in the approximation guarantee. This is a promising first step towards designing simple truthful deterministic mechanisms with non-trivial approximation guarantees. In contrast to algorithms using the time-intensive solution of convex optimization problems, such quick and simple greedy rules are much more suitable for application in practice. Providing good and simple mechanisms is a major open direction for future work.

Theorem 17. *Algorithm 6 is deterministic and monotone. The computed solution is a $O(\rho \cdot \log n)$ -*

Algorithm 6: Monotone $O(\rho \cdot \log n)$ -algorithm for Maximum Weighted Independent Set.

```

1 Sort the set of bids  $B = \{b_v \mid v \in V\}$  in decreasing order, let  $b_i$  be the  $i$ -th highest bid;
2 for  $i = 1$  to  $n$  do
3   Let  $V_i = \{v \in V \mid b_v \geq b_i\}$  and  $S_i = \emptyset$ ;
4   for  $v \in V_i$  in increasing order of  $\pi$  values do
5      $\lfloor$  If  $N(v) \cap S_i = \emptyset$ , add  $v$  to  $S_i$ 
6 Output  $S = S_{i^*}$  with  $i^* = \arg \max_i \sum_{v \in S_i} b_v$ ;

```

approximation for the maximum weight independent set problem.

Proof. We first prove that the algorithm is monotone. We show that if $v \notin S$ and lies a value $b'_v < b_v$, then v will never be able to become part of S . Suppose v is currently first considered in iteration i . Submitting a smaller bid causes v to be considered at a later point $j > i$. In the sets V_i, \dots, V_{j-1} player v is replaced by a different player, sets V_1, \dots, V_{i-1} and V_j, \dots, V_n remain as before, and so do S_1, \dots, S_{i-1} and S_j, \dots, S_n . If one of these sets was chosen as the best set before, then v will again not be part of S if he lies. The only sets that can be different now are S_i, \dots, S_{j-1} , in which v cannot be present. If previously set S_k with $k \in \{i, \dots, j-1\}$ was chosen as the best set, it did not include v . Thus, in the run v was blocked by some other vertex. Removing v does not change the execution of the algorithm, thus the same set will be computed again – however, due to the change in the ordering it will now appear as S_{k-1} . The only sets S_k that can change are the ones with $k \in \{i, \dots, j-1\}$ where v was included before. However, if a new optimal set appears here, it does not include v as well. In conclusion, if $v \notin S$, he cannot become included into S by reducing his bid.

To bound the approximation factor, we use an argument similar to [18]. Let us consider the problem on the subset V_i and assume all vertices have value b_i . For this problem, our algorithm is equivalent to the greedy ρ -approximation algorithm for unweighted vertices. Hence, for this subproblem we obtain a ρ -approximation. With S'_i being the optimum for this subproblem, then we have

$$\sum_{v \in S} b_v = \max_{i=1}^n \sum_{v \in S_i} b_v \geq \max_{i=1}^n \{|S_i| \cdot b_i\} \geq \frac{1}{\rho} \cdot \max_{i=1}^n \{|S'_i| \cdot b_i\}.$$

Now consider intervals $I_j = (b_1/2^j, b_1/2^{j-1}]$, for $j = 1, \dots, \log n$. The last interval we set $I_{(\log n)+1} = [0, b_1/n]$. For each such interval we consider the subgraph of vertices v with value $b_v \in I_j$ and the optimum solution S^j w.r.t. to this subinstance. Consider all i such that $b_i \in I_j$. It is easy to see that for all $j = 1, \dots, \log n$

$$\frac{1}{\rho} \cdot \max_{i: b_i \in I_j} \{|S'_i| \cdot b_i\} \geq \frac{1}{2 \cdot \rho} \sum_{v \in S^j} b_v.$$

For $j = (\log n) + 1$ we obviously have $|S'_1| \cdot b_1 \geq \sum_{v \in S^j} b_v$. Thus, in total we have

$$\frac{1}{\rho} \cdot \max_{i=1}^n \{|S'_i| \cdot b_i\} \geq \frac{1}{2 \cdot \rho \cdot \log n + \rho} \cdot \sum_{j=1}^{\log n} \sum_{v \in S^j} b_v \geq \frac{1}{2 \cdot \rho \cdot \log n + \rho} \cdot \sum_{v \in S^*} b_v,$$

since the sum of values for the optimal solutions in the intervals is bigger than the global optimum S^* . This proves the approximation factor. \square

This represents a promising first step towards designing simple truthful deterministic mechanisms with non-trivial approximation guarantees. In contrast to algorithms using the time-intensive solution of convex optimization problems, such quick and simple greedy rules are much more suitable for application in practice. In addition, the concept of truthfulness in expectation used in the previous sections has drawbacks, e.g., it is not enough to motivate *risk-aware* bidders to reveal their valuations truthfully. While there are many open problems stemming from our work (e.g., improving the approximation bounds for specific interference models), providing good and simple mechanisms for stronger notions of truthfulness is a challenging and arguably the most interesting avenue for future work.

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